

# Spectroscopy of a weakly isolated horizon

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## Abstract

The spectroscopy of a weakly isolated horizon (WIH) has been investigated. We obtain an equally spaced entropy spectrum with its quantum equal to the one given by Bekenstein [5]. We demonstrate that the quantization of entropy and area is a generic property of horizon, and the results exit in a wide class of spacetimes admitting weakly isolated horizons. Our results also indicate that the entropy quantum of the black hole horizon is closely related to Hawking temperature.

**Keywords:** weakly isolated horizon, quantization, entropy spectrum

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## I. INTRODUCTION

Since the first exact solution of Einstein equation was found out, studying black holes' properties has become an important part of gravitational physics. Properties of black holes, for example, laws of black hole mechanics, Hawking radiation and black hole spectroscopy, caused deep, unsuspected connections among classical general relativity, quantum physics and statistical mechanics. However, the traditional definition of a black hole [1], is too global and idealized: it requires knowledge of the entire future of the space-time, so it is often cumbersome to use for the requirements of practical research [2]. In recent years, a new, quasi-local framework was introduced by Ashtekar and his collaborators to analyze different facets of black holes in a unified way [2–4]. Compared with the event horizon, this framework doesn't need the knowledge of overall space-time, and only involves quasi-local conditions, so it accords with the practical physical process. In this framework, black holes in equilibrium (no matter and energy flow across the horizon) are described by weakly isolated horizons (WIH).

In 1970s, Bekenstein proved that the quantum of the black hole horizon is given as  $(\Delta A)_{min} = 8\pi l_p^2$  [5]. From then on, there has been much attention paid to the quantization of black hole entropy spectrum and area spectrum [6–15], and many methods rely on the quasi-normal frequency which requires the knowledge of the global geometry of the space-time, not just the geometry near a horizon. Recently, Majhi and Vagenas [16] proposed a new approach to derive the entropy spectrum and the horizon area quantum utilizing solely the periodicity of imaginary time and the Bohr-Sommerfeld quantization rule, and there was no use at all of the quasinormal frequencies to obtain the result. Later on, there were many works using this method to study the entropy spectrum and area spectrum of a wide variety of spacetimes [17]. We use a similar method that does not need the periodicity of imaginary time, and our method could apply to more general spacetimes. In this paper, we quantize the spectroscopy of a weakly isolated horizon which is a locally defined black hole and already contains all the stationary black holes. Our method also shows that there exists close relationship between the entropy quantum and Hawking temperature, which is an interesting thing.

This paper is organized as follows. In section 2, we briefly review the definition of the WIH and the geometry near it. In section 3, we quantize a weakly isolated horizon. Finally,

the conclusions are given in section 4.

## II. GEOMETRY OF A WEAKLY ISOLATED HORIZON

In this section we will briefly review some geometric properties of WIH [2–4]. As in Refs. [18], it is very convenient to introduce the Bondi-like coordinates  $(u, r, \theta, \varphi)$ , which are well defined on the horizon, and choose a set of null tetrad, which satisfy Bondi gauge, to study the behavior in the neighborhood of WIH. The null tetrad can be expressed as

$$\begin{aligned} l^a &= \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \varsigma} + \bar{X} \frac{\partial}{\partial \bar{\varsigma}} \\ n^a &= -\frac{\partial}{\partial r} \\ m^a &= \omega \frac{\partial}{\partial r} + \xi_3 \frac{\partial}{\partial \varsigma} + \xi_4 \frac{\partial}{\partial \bar{\varsigma}} \\ \bar{m}^a &= \bar{\omega} \frac{\partial}{\partial r} + \bar{\xi}_3 \frac{\partial}{\partial \bar{\varsigma}} + \bar{\xi}_4 \frac{\partial}{\partial \varsigma}, \end{aligned} \quad (1)$$

where  $U \hat{=} X \hat{=} \omega \hat{=} 0$  on the horizon  $H$  (following the notation in Ref. [2], equalities restricted to  $H$  will be denoted by “ $\hat{=}$ ”), and  $\varsigma = e^{i\phi} \cot \frac{\theta}{2}$ . Note that  $n^a$  and  $l^a$  are future directed. We take the spacetime metric  $g_{ab}$  to have a signature  $(-, +, +, +)$ , so the metric can be expressed as

$$g_{ab} = m_a \bar{m}_b + \bar{m}_a m_b - n_a l_b - l_a n_b. \quad (2)$$

The definition of WIH [2–4] implies that there is a one form  $\omega_a$  on  $H$  which satisfy the following relationship,  $\mathcal{L}_l \omega^a \hat{=} 0$  and  $D_a l^b \hat{=} \omega_a l^b$ , where  $D_a$  is the induced covariant derivative on  $H$ . In terms of the Newman-Penrose formalism,  $\omega_a$  can be explicitly expressed as

$$\omega_a = -(\varepsilon + \bar{\varepsilon})n_a + (\alpha + \bar{\beta})\bar{m}_a + (\bar{\alpha} + \beta)m_a = -(\varepsilon + \bar{\varepsilon})n_a + \pi\bar{m}_a + \bar{\pi}m_a, \quad (3)$$

which means  $(\varepsilon + \bar{\varepsilon})$  is constant on  $H$  from  $\mathcal{L}_l \omega^a \hat{=} 0$ . The commutators of the null tetrad  $[l^a, n^a]$  and  $[m^a, n^a]$  tell us that

$$\begin{aligned} \frac{\partial U}{\partial r} &= (\varepsilon + \bar{\varepsilon}) + \bar{\pi}\bar{\omega} + \pi\omega, \quad \frac{\partial X}{\partial r} = \bar{\pi}\bar{\xi}_4 + \pi\xi_3, \quad \frac{\partial \omega}{\partial r} = \bar{\pi} + \bar{\lambda}\bar{\omega} + \mu\omega, \\ \frac{\partial \xi_3}{\partial r} &= \bar{\lambda}\bar{\xi}_4 + \mu\xi_3, \quad \frac{\partial \xi_4}{\partial r} = \bar{\lambda}\bar{\xi}_3 + \mu\xi_4, \end{aligned} \quad (4)$$

which means  $\frac{\partial U}{\partial r} \hat{=} (\varepsilon + \bar{\varepsilon})$ . Then the behavior of functions  $U$ ,  $X$  and  $\omega$  near  $H$  is

$$\begin{aligned} U &= (\varepsilon + \bar{\varepsilon})r + O(r^2), \\ X &= O(r), \omega = O(r). \end{aligned} \quad (5)$$

Using Eq. (1), Eqs. (2) and Eq. (5), the out-going null geodesic can be calculated as

$$0 = 2du^2 \frac{\bar{\xi}_3 X - \bar{\xi}_4 \bar{X}}{|\xi_4|^2 - |\xi_3|^2} \times \frac{\xi_3 \bar{X} - \xi_4 X}{|\xi_4|^2 - |\xi_3|^2} - 2du^2 \left( U - \frac{\xi_4 \bar{\omega} - \bar{\xi}_3 \omega}{|\xi_4|^2 - |\xi_3|^2} X - \frac{\bar{\xi}_4 \omega - \xi_3 \bar{\omega}}{|\xi_4|^2 - |\xi_3|^2} \bar{X} \right) + 2dudr, \quad (6)$$

which leads to

$$\frac{dr}{du} = U. \quad (7)$$

Based on Ref. [2], not any choice of time direction can give a Hamiltonian evolution, and only some suitably chosen time direction can lead to a well-defined horizon mass. In Ref. [2], A. Ashtekar and B. Krishnan gave a canonical way to choose the time direction  $t^a$  for a WIH, and the restriction of  $t^a$  to  $H$  should be a linear combination of a null normal  $l^a$  and the axisymmetric vector  $\psi^a$ ,

$$t^a \hat{=} B_t l^a - \Omega_t \psi^a, \quad (8)$$

where  $B_t$  and  $\Omega_t$  are constant on the horizon. Compared with the Schwarzschild case, the parameter of  $t^a$  takes the place of the Killing time. Using Eq. (7) and Eq. (8), we get the time derivative of  $r$  along the outgoing geodesic,

$$\dot{r} = \frac{du}{dt} \frac{dr}{du} = (B_t + O(r))U = B_t(\varepsilon + \bar{\varepsilon})r + O(r^2). \quad (9)$$

With the canonical time direction  $t^a$ , A. Ashtekar and B. Krishnan [2] established the zeroth and the first law of WIH. By definition, the surface gravity of  $H$  is  $\kappa_t := B_t l^a \omega_a = B_t(\varepsilon + \bar{\varepsilon})$ . Because  $B_t(\varepsilon + \bar{\varepsilon})$  is constant on  $H$ , the zeroth law of black hole mechanics is valid for WIH. The first law is expressed as

$$\delta M_H^{(t)} = \frac{\kappa_t}{8\pi} \delta a_H + \Omega_t \delta J_H, \quad (10)$$

where  $M_H^{(t)}$  is the horizon mass,  $a_H$  is the area of the cross section of WIH,  $\Omega_t$  is the angular velocity of the horizon and  $J_H = -\frac{1}{8\pi} \oint_S (\omega_a \psi^a) dS$  is the angular momentum. The first law of WIH is the generalization of the first law of stationary black holes. Because Refs. [19, 20] studied the Hawking radiation of a WIH, so the first law of WIH mechanics is upgraded to the first law of WIH thermodynamics which can be expressed as

$$\delta M_H^{(t)} = T_H \delta S_H + \Omega_t \delta J_H, \quad (11)$$

where  $S_H = \frac{a_H}{4}$  is the entropy of WIH, and  $T_H = \frac{\kappa_t}{2\pi}$  is the Hawking temperature.

### III. QUANTIZATION OF A WEAKLY ISOLATED HORIZON

In this section, we quantize a weakly isolated horizon. We consider a tunneling process cross the WIH, and investigate an action of the form

$$I = \int p_i dq_i = \int \int dp_i dq_i = \int \int \frac{d\varepsilon}{\dot{q}_i} dq_i, \quad (12)$$

where  $p_i$  is the conjugate momentum of the coordinate  $q_i$  with  $i = 0, 1$  for which  $q_0 = \tau$  and  $q_1 = r$ . Note that we use the Euclidean time  $q_0 = \tau$  and the Einstein summation convention. To get the last equation, we have used Hamilton's equation  $\dot{q}_i = \frac{d\varepsilon}{dp_i}$ , where the Hamiltonian  $\varepsilon$  is the energy of the emitted particle. Write Eq. (12) explicitly,

$$I = \int p_i dq_i = \int \int d\varepsilon d\tau + \int \int \frac{d\varepsilon}{\dot{r}} dr. \quad (13)$$

We shall obtain the quantity  $\dot{r}$  that appears in Eq. (13). Let us consider the radial null path, and our analysis will concentrates on the outgoing path, since these is the one related to the quantum mechanically nontrivial features [21]. Since in Eq. (13)  $\tau$  is the Euclidean time, we substitute the transformation  $t \rightarrow i\tau$  into Eq. (9) and get the outgoing radial null path. This leads to

$$\dot{r} \equiv \frac{dr}{d\tau} = iB_t(\varepsilon + \bar{\varepsilon})r + O(r^2) = R_+(r). \quad (14)$$

Now, using Eq. (14), we get

$$\int \int d\varepsilon d\tau = \int \int d\varepsilon \frac{dr}{R_+(r)} = \int \int d\varepsilon \frac{dr}{\dot{r}}, \quad (15)$$

so the action (13) reads

$$I = \int p_i dq_i = 2 \int \int d\varepsilon d\tau = 2 \int \int d\varepsilon \frac{dr}{\dot{r}}. \quad (16)$$

Putting Eq. (14) into the above equation, and integrating around the pole at the horizon, that is,  $r = 0$ , we get

$$\begin{aligned} I &= \int p_i dq_i = 2 \int \int d\varepsilon \frac{dr}{\dot{r}} \\ &= 2 \int \int_{r_{in}}^{r_{out}} \frac{dr}{iB_t(\varepsilon + \bar{\varepsilon})r + O(r^2)} d\varepsilon \\ &= 2\pi \int \frac{d\varepsilon}{B_t(\varepsilon + \bar{\varepsilon})} = -2\pi \int \frac{dM_H}{\kappa_t}, \end{aligned} \quad (17)$$

where we have used the relationship of energy conservation  $dM_H = -d\varepsilon$  in the last equation, and  $\kappa_t = B_t(\varepsilon + \bar{\varepsilon})$  is the surface gravity of the WIH. So the action we considered is

$$I = \int p_i dq_i = -2\pi \int \frac{dM_H}{\kappa_t}. \quad (18)$$

As we all know that the temperature of a black hole is proportional to the surface gravity of the horizon

$$T_H = \frac{\hbar \kappa_t}{2\pi}, \quad (19)$$

thus Eq. (18) becomes

$$I = \int p_i dq_i = -\hbar \int \frac{dM_H}{T_H} = -\hbar \Delta S_H, \quad (20)$$

where we have used the first law of WIH thermodynamics (11) with  $\Omega_t = 0$  in the last step, and  $\Delta S_H < 0$  is the change of entropy of WIH after emission of a particle.

At last, implementing the Bohr-Sommerfeld quantization rule

$$\int p_i dq_i = nh \quad (21)$$

in Eq. (20), we derive the WIH entropy spectrum

$$|\Delta S_H| = 2\pi n, \quad (22)$$

where  $n = 1, 2, 3, \dots$ , and it is straightforward to see that the minimum spacing in the entropy is given by

$$|\Delta S_H| = 2\pi. \quad (23)$$

Thus, the entropy spectrum is quantized and equidistant for a weakly isolated horizon. Recalling that in the framework of Einstein's theory of gravity, black hole entropy is proportional to the black hole horizon area [5],  $S_H = \frac{A}{4l_p^2}$ . It is evident that if we employ the spacing of the entropy spectrum given in Eq. (23), the quantum of the WIH area has the form

$$\Delta A = 8\pi l_p^2, \quad (24)$$

which is the same as the area quantum derived by Bekenstein [5].

For an axial symmetric horizon, using the method in Ref. [23], the last equation of (16) can be modified as

$$\begin{aligned}
I &= 2\left[\int \int d\varepsilon \frac{dr}{\dot{r}} - \int p_\phi d\phi\right] = 2\int \int \frac{d\varepsilon - \dot{\phi} dp_\phi}{\dot{r}} dr \\
&= -2\int \int \frac{dM_H - \Omega_t dJ_H}{\dot{r}} dr = -2\pi \int \frac{dM_H - \Omega_t dJ_H}{B_t(\varepsilon + \bar{\varepsilon})} = -\hbar \Delta S_H.
\end{aligned} \tag{25}$$

$p_\phi$  is the angular momentum of the emitted particle, and we have used the conservation of angular momentum, that is,  $dp_\phi = -dJ_H$  and  $\dot{\phi} = \Omega_t$  in the third equation. In the last equality, we have used Eq. (19) and the first law of WIH thermodynamics (11). The result is the same as that of the non-rotating WIH (20).

Let us give some discussions. Firstly, Although our method is similar to the method used in Refs. [16, 17], there exists a significant difference. The derivation of Refs. [16, 17] relies on the periodicity of imaginary time, while we do not need it. Secondly, We consider a tunneling process cross the WIH, and concentrate on the outgoing path. From the calculation, it is easy to see that the ingoing path does not contribute to the action (12). Thirdly, our method considers a tunneling process and uses the technology of the tunneling method [21, 22] which has been successfully used to calculate Hawking temperature of a variety of spacetimes, so our method has more generality. Finally, our method also shows that there exists close relationship between the entropy quantum and Hawking temperature, which reflects to some extent the viewpoint of the emergent perspective of gravity that temperature means existence of underlying degrees of freedom [24].

#### IV. CONCLUSIONS

In this paper, we have quantized the entropy and the area of a weakly isolated horizon, and obtain the quantized entropy and area spectrum which are the same as Bekenstein's original results [5]. Our results indicate that the quantization of entropy of the black hole horizon is a generic property of horizon, and is closely related to Hawking temperature.

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